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CLASS GROUPS AND PICARD GROUPS OF NORMAL SCHEMES

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Let Y be a normal reduced noetherian scheme with singular set S. Let $X \xrightarrow{\phi} Y$ be a dominant map of irreducible noetherian schemes such that $T = \phi^{-1}(S)$ has codimension ≥ 2 in X, T contains the singular set of X, and $\phi \mid T : T \to S$ is an isomorphism of schemes. For example, X could be a localization or completion of Y along S.

Theorem. *The map* $\ker(\mathrm{Pic}(Y) \to \mathrm{Pic}(X)) \to \ker(\mathrm{Cl}(Y) \to \mathrm{Cl}(X))$ *is an isomorphism.*

As corollaries, we can immediately recover some standard results:

Corollary 1 (Mori's Theorem [2, 6.12]). *Let A be a ring and $\mathscr{M}$ an ideal contained in the Jacobson Radical of A. Let $\hat{A}$ be the completion of A at $\mathscr{M}$. If $\hat{A}$ is normal, then* $\mathrm{Cl}(A) \to \mathrm{Cl}(\hat{A})$ *is a monomorphism.*

Corollary 2 ([2, 18.6]). *Let $\mathscr{M}_1, \ldots, \mathscr{M}_n$ be the maximal ideals at which the normal domain A is not factorial. Let $S = A - (\bigcup_{i=1}^{n} \mathscr{M}_i)$. Then* $\mathrm{Pic}(A)$ *is precisely the kernel of the map* $\mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$.

The theorem follows from the classification of *Milnor Patching Diagrams.* A commutative square

$$\begin{array}{ccc} V & \xrightarrow{f} & X \\ \downarrow & & \downarrow \delta \\ U & \xrightarrow[g]{} & Y \end{array} \qquad (1)$$

is called a Milnor Patching Diagram if $\mathbf{VB}(Y) = \mathbf{VB}(X) \times_{\mathbf{VB}(V)} \mathbf{VB}(U)$, where $\mathbf{VB}(Y)$ is the category of vector bundles over Y. (Recall that if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are categories and we are given functors $F:\mathbf{A}\to\mathbf{C}$, $G:\mathbf{B}\to\mathbf{C}$, then the fiber product category $\mathbf{A}\times_{\mathbf{C}}\mathbf{B}$ is defined to be the category whose objects are triples $(A,B,\alpha:FA\to GB)$ with $A\in\mathrm{Ob}(\mathbf{A})$, $B\in\mathrm{Ob}(\mathbf{B})$ and $\alpha\in\mathrm{Arr}(\mathbf{C})$ an isomorphism. An arrow $(A,B,\alpha)\to(A',B',\alpha')$ is a pair of arrows $A\to A'$ and $B\to B'$ making the obvious diagram commute.)

In [3] and [4], we classified Milnor Patching Diagrams in which all of the schemes were affine. In [5], we will extend these results to more general schemes. The trick in all of this work is to reduce to the following lemma, which will also suffice to yield the theorem announced above.

Milnor Patching Lemma. *Suppose that Y is normal, X is irreducible, $\delta:X\to Y$ is dominant, and $g:U\to Y$ is an open inclusion. Put $S=Y-U$ and $T=\delta^{-1}(S)$. Suppose that $\delta|_T:T\to S$ is an isomorphism. Then* (1) *is a Milnor Patching Diagram.*

Proof. When X and Y are affine and δ is flat, this is Theorem 2.4 of [4]. More generally, if X is both affine and either flat or of finite type over Y, it is a special case of Theorem 1 of [5]. (The latter also corrects a slight misstatement in the hypotheses used in [4].) To do the general case, note that the theorem is local on Y, so we may assume that Y is affine. The assumptions guarantee that if $\hat{X}$ is the completion of X along T and $\hat{Y}$ is the completion of Y along X, then $\hat{X}\to\hat{Y}$ is an isomorphism; in particular X is affine. Now consider the two diagrams

$$\begin{array}{ccc} U\times_Y\hat{X} & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V\times_X\hat{X} & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

By Lemma 1.6 of [3] (and it is reasonably straightforward in any case) it suffices to deal with each of these diagrams separately. In each case δ has been replaced by a flat map. In the first case, $\hat{X}$ and Y are both affine, so we are done. In the second case, the map $\hat{X}\to X$ is affine, so we can work locally on X and we are done. □

Now to prove the theorem announced at the beginning of this note, set $U=Y-S$, $V=X-T$, and notice that

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \end{array}$$

satisfies the conditions to be a Milnor Patching diagram. Since $\mathrm{Cl}(Y)=\mathrm{Pic}(U)$ ([1, 21.6.12] or [2, 18.7]), an element of $\ker(\mathrm{Cl}(Y)\to\mathrm{Cl}(X))$ can be represented by a line bundle $\mathscr{L}$ over U that pulls back to a trivial bundle $\mathscr{T}$ over V. Let $\mathscr{T}'$ be a trivial bundle over X of the same rank and choose any isomorphism $\alpha : \mathscr{T} \to f^*\mathscr{T}'$. Then the data $(\mathscr{L}, \mathscr{T}', \alpha)$ define a line bundle $\mathscr{L}' \in \mathrm{Pic}(Y)$ such that $\mathscr{L}'$ pulls back to $\mathscr{T}'$ over X (i.e., $\mathscr{L}' \in \ker(\mathrm{Pic}(Y) \to \mathrm{Pic}(X))$) and $\mathscr{L}'$ restricts to $\mathscr{L} \in \mathrm{Cl}(Y)$.

This demonstrates the surjectivity of the map in question; injectivity follows from the injectivity of the map Pic→Cl, which is standard ([1, 21.6.10] or [2, 18.4]). □

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